

The Baker-Campbell-Hausdorff Formula and the Zassenhaus Formula in Synthetic Differential Geometry

Hirokazu NISHIMURA
Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki, 305-8571, JAPAN

June 3, 2013

Abstract

After the torch of Anders Kock [Taylor series calculus for ring objects of line type, Journal of Pure and Applied Algebra, 12 (1978), 271-293], we will establish the Baker-Campbell-Hausdorff formula as well as the Zassenhaus formula in the theory of Lie groups.

1 Introduction

The Baker-Campbell-Hausdorff formula (the BCH formula for short) was first discovered by Campbell ([2] and [3]) on the closing days of the 19th century so as to construct a Lie group directly from a given Lie algebra (i.e., Lie's third fundamental theorem !). However, his investigation failed in convergence problems, let alone dealing only with matrix Lie algebras. The BCH formula was finally established by Baker [1] and Hausdorff [6] independently within a somewhat more abstract framework of formal power series on the dawning days of the 20th century, getting rid of convergence problems completely while losing touch with the theory of Lie groups. The BCH formula resurrected its touch with the theory of Lie groups thanks to Magnus [10] in the middle of the 20th century.

The BCH formula claims, roughly speaking, that the multiplication in a Lie group is already encoded in its Lie algebra. More precisely, the multiplication in a Lie group is expressible in terms of Lie brackets in its Lie algebra, which readily gives rise to Lie's second fundamental theorem in the theory of finite-dimensional Lie groups, though the modern treatment of the theory of finite-dimensional Lie

groups is liable to base Lie's second fundamental theorem somewhat opaquely upon the Frobenius theorem.

The so-called Taylor formula was introduced by the English mathematician called Brook Taylor in the early 18th century, though its pedigree can be traced back even to Zeno in ancient Greece. Kock [7] has shown that the nature of the Taylor formula in differential calculus is more combinatorial or algebraic than analytical, dodging convergence problems completely, as far as we are admitted to speak on the infinitesimal level, where nilpotent infinitesimals are available in plenty. The principal objective in this paper is to do the same thing to the BCH formula and its inverse companion called the Zassenhaus formula in the theory of Lie groups, though we must confront the *noncommutative* world in sharp contrast to the Taylor formula living a *commutative* life. We have found out that the Zassenhaus formula is much easier to deal with than the BCH formula itself, albeit, historically speaking, the former having been found out by Zassenhaus [20] within an abstract framework of formal power series more than three decades later than the latter and its continuous counterpart having been established by Fer [4] four years later than [10]. Quirky enough, our BCH formula diverges from the usual one in the 4-th order. The BCH formula will be dealt with in §7 and §8 by two different methods, while we will be engaged upon the Zassenhaus formula in §6. We approach the BCH formula in anticipation of its validity in §7 by using only the left logarithmic derivative of the exponential mapping, while we will do so from scratch in §8 by using both of the left and right logarithmic derivatives of the exponential mapping. As is expected, the latter proofs are longer than the former ones.

We will work within the framework of synthetic differential geometry as in [9]. We assume the reader to be familiar with Chapters 1-3 of [9]. Now we fix our terminology and notation. Given a microlinear space M , we denote M^D by \mathbf{TM} , while we denote the tangent space of M at $x \in M$ by $\mathbf{T}_x M = \{\gamma \in \mathbf{TM} \mid \gamma(0) = x\}$. Given a mapping $f : M \rightarrow N$ of microlinear spaces, its differential is denoted by $\mathbf{d}f$, which is a mapping from \mathbf{TM} to \mathbf{TN} , assigning $f \circ \gamma \in \mathbf{TN}$ to each $\gamma \in \mathbf{TM}$. We denote the identity mapping of M by id_M . The unit element of a group G is usually denoted by e . In the proof of a theorem or the like, we insert some comment surrounded with parentheses).

2 The Lie Algebra of a Lie Group

Definition 1 *A Lie group is a group which is microlinear as a space.*

Notation 2 *Given a Lie group G , its tangent space $\mathbf{T}_e G$ at e is usually denoted by its corresponding German letter \mathfrak{g} .*

From now on, G will always be assumed to be a Lie group with $\mathfrak{g} = \mathbf{T}_e G$.

Proposition 3 *Given $X \in \mathfrak{g}$ and $(d_1, d_2) \in D(2)$, we have*

$$X_{d_1+d_2} = X_{d_1} \cdot X_{d_2}$$

Proof. By the same token as in Proposition 3, §3.2 of [9]. ■

Corollary 4

$$X_{-d} = (X_d)^{-1}$$

Proof. Evidently

$$(d, -d) \in D(2)$$

obtains, so that we get

$$e = X_{d+(-d)} = X_d \cdot X_{-d} = X_{-d} \cdot X_d$$

by the above proposition. ■

Proposition 5 *Given $X, Y \in \mathfrak{g}$ and $d \in D$, we have*

$$(X + Y)_d = X_d \cdot Y_d = Y_d \cdot X_d$$

Proof. By the same token as in Proposition 6, §3.2 of [9]. ■

Theorem 6 *Given $X, Y \in \mathfrak{g}$, there exists a unique $Z \in \mathfrak{g}$ with*

$$X_{d_1} \cdot Y_{d_2} \cdot X_{-d_1} \cdot Y_{-d_2} = Z_{d_1 d_2}$$

for any $d_1, d_2 \in D$.

Proof. By the same token as in pp.71-72 of [9]. ■

Definition 7 *We denote Z in the above theorem by $[X, Y]$, so that we have a function*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

called the Lie bracket.

Theorem 8 *The \mathbb{R} -module \mathfrak{g} endowed with the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra.*

Proof. By the same token as in Proposition 7 (§3.2) of [9]. ■

Proposition 9 *Given a homomorphism*

$$\varphi : G \rightarrow H$$

of Lie groups, the mapping

$$\varphi' : \mathfrak{g} \rightarrow \mathfrak{h}$$

obtained as the restriction of the differential

$$\mathbf{d}\varphi : \mathbf{T}G \rightarrow \mathbf{T}H$$

to $\mathfrak{g} = \mathbf{T}_e G$ is a homomorphism of Lie algebras.

Proof. Given $X, Y \in \mathfrak{g}$ and $d_1, d_2 \in D$, we have

$$\begin{aligned}
& (\mathbf{d}\varphi([X, Y]))_{d_1 d_2} \\
&= \varphi([X, Y]_{d_1 d_2}) \\
&= \varphi(X_{d_1} \cdot Y_{d_2} \cdot X_{-d_1} \cdot Y_{-d_2}) \\
&= \varphi(X_{d_1}) \cdot \varphi(Y_{d_2}) \cdot \varphi(X_{-d_1}) \cdot \varphi(Y_{-d_2}) \\
&= (\mathbf{d}\varphi(X))_{d_1} \cdot (\mathbf{d}\varphi(Y))_{d_2} \cdot (\mathbf{d}\varphi(X))_{-d_1} \cdot (\mathbf{d}\varphi(Y))_{-d_2} \\
&= [\mathbf{d}\varphi(X), \mathbf{d}\varphi(Y)]_{d_1 d_2}
\end{aligned}$$

so that φ' preserves Lie brackets. ■

The succeeding simple lemma will be useful in the last section.

Lemma 10 *Given $X, Y \in \mathfrak{g}$, we have*

$$[X, [Y, [X, Y]]] = [Y, [X, [X, Y]]]$$

Proof. This follows easily from the following Jacobi identity:

$$[X, [Y, [X, Y]]] + [Y, [[X, Y], X]] + [[X, Y], [X, Y]] = 0$$

■

Notation 11 *Given a Euclidean \mathbb{R} -module V which is microlinear as a space, the totality of bijective homomorphisms of \mathbb{R} -modules from V onto itself is denoted by $GL(V)$, which is a Lie group with composition of mappings as its group operation (cf. Proposition 5 (§§3.2) of [9]). Its Lie algebra is usually denoted by $\mathfrak{gl}(V)$.*

Proposition 12 *Given a Euclidean \mathbb{R} -module V which is microlinear as a space, the Lie algebra $\mathfrak{gl}(V)$ can naturally be identified with the Lie algebra of homomorphisms of \mathbb{R} -modules from V into itself with its Lie bracket*

$$[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$$

for any homomorphisms φ, ψ of \mathbb{R} -modules from V into itself.

Proof. Given a mapping $X : D \rightarrow GL(V)$ with $X_0 = \text{id}_V$, there exists a unique mapping $\varphi : V \rightarrow V$ such that

$$X_d(u) = u + d\varphi(u)$$

for any $d \in D$ and any $u \in V$, since the \mathbb{R} -module V is Euclidean by assumption. Since $X_d \in GL(V)$, we have

$$\alpha u + d\varphi(\alpha u) = X_d(\alpha u) = \alpha X_d(u) = \alpha u + \alpha d\varphi(u)$$

for any $\alpha \in \mathbb{R}$, any $u \in V$ and any $d \in D$, so that we get

$$\varphi(\alpha u) = \alpha \varphi(u)$$

for any $\alpha \in \mathbb{R}$ and any $u \in V$, which implies that the mapping $\varphi : V \rightarrow V$ is a homomorphism of \mathbb{R} -modules (cf. Proposition 10 (§§1.2) in [9]). Conversely, given a homomorphism φ of \mathbb{R} -modules from V into itself and $d \in D$, $\text{id}_V + d\varphi$ is obviously a homomorphism of \mathbb{R} -modules from V into itself, and we have

$$(\text{id}_V + d\varphi) \circ (\text{id}_V - d\varphi) = (\text{id}_V - d\varphi) \circ (\text{id}_V + d\varphi) = \text{id}_V$$

so that the mapping $\text{id}_V + d\varphi$ is bijective. Therefore we are sure that the \mathbb{R} -module $\mathfrak{gl}(V)$ is naturally identified with the \mathbb{R} -module of homomorphisms of \mathbb{R} -modules from V into itself. It remains to show that this identification preserves Lie brackets. Let us assume that $X \in \mathfrak{gl}(V)$ corresponds to the homomorphism φ of \mathbb{R} -modules from V into itself, while $Y \in \mathfrak{gl}(V)$ corresponds to the homomorphism ψ of \mathbb{R} -modules from V into itself. Then, given $d_1, d_2 \in D$, we have

$$\begin{aligned} & [X, Y]_{d_1 d_2} \\ &= X_{d_1} \cdot Y_{d_2} \cdot X_{-d_1} \cdot Y_{-d_2} \\ &= (\text{id}_V + d_1 \varphi) \circ (\text{id}_V + d_2 \psi) \circ (\text{id}_V - d_1 \varphi) \circ (\text{id}_V - d_2 \psi) \\ &= \{\text{id}_V + d_1 \varphi + d_2 \psi + d_1 d_2 \varphi \circ \psi\} \circ \{\text{id}_V - d_1 \varphi - d_2 \psi + d_1 d_2 \varphi \circ \psi\} \\ &= \text{id}_V - d_1 \varphi - d_2 \psi + d_1 d_2 \varphi \circ \psi + d_1 \varphi - d_1 d_2 \varphi \circ \psi + d_2 \psi - d_1 d_2 \psi \circ \varphi + d_1 d_2 \varphi \circ \psi \\ &= \text{id}_V + d_1 d_2 (\varphi \circ \psi - \psi \circ \varphi) \end{aligned}$$

so that our identification of $\mathfrak{gl}(V)$ with the \mathbb{R} -module of homomorphisms of \mathbb{R} -modules from V into itself indeed preserves Lie brackets. ■

3 The Adjoint Representations

Notation 13 Given $x \in G$, the mapping $y \in G \mapsto xyx^{-1} \in G$ is obviously a homomorphism of groups, naturally giving rise to a mapping $\mathfrak{g} \rightarrow \mathfrak{g}$ as derivation, which we denote by $\text{Ad} x \in GL(\mathfrak{g})$. Thus we have a homomorphism $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ of groups, naturally giving rise to a mapping $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ as derivation.

Theorem 14 Given $X, Y \in \mathfrak{g}$, we have

$$(\text{ad } X)(Y) = [X, Y]$$

Proof. Given $d, d' \in D$, we have

$$\begin{aligned} & ((\text{Ad } X_d)(Y) - Y)_{d'} \\ &= X_d \cdot Y_{d'} \cdot X_{-d} \cdot Y_{-d'} \\ & \text{By Proposition 5(} \\ &= [X, Y]_{dd'} \\ &= (d[X, Y])_{d'} \end{aligned}$$

so that we have the desired formula. ■

4 The Exponential Mapping

Our notions of a one-parameter subgroup, a left-invariant vector field, etc. are standard, and it is easy to see that

Proposition 15 *Given a mapping $\theta : \mathbb{R} \rightarrow G$, the following conditions are equivalent:*

1. *The mapping $\theta : \mathbb{R} \rightarrow G$ is a one-parameter subgroup.*
2. *The mapping $\theta : \mathbb{R} \rightarrow G$ is a flow of a left invariant vector field on G with $\theta(0) = e$.*
3. *The mapping $\theta : \mathbb{R} \rightarrow G$ is a flow of a right invariant vector field on G with $\theta(0) = e$.*

Notation 16 *Given $X \in \mathfrak{g}$, if there is a one-parameter subgroup $\theta : \mathbb{R} \rightarrow G$ with $d\theta(i_D^{\mathbb{R}}) = X$, then we write $\exp^G X$ or $\exp X$ for $\theta(1)$.*

The following definition is borrowed from 38.4 in [8], which is, in turn, owing to the research [13]-[18] of Omori et al.

Definition 17 *A Lie group G is called regular provided that, for any mapping $\varsigma : \mathbb{R} \rightarrow \mathfrak{g}$, there exists a mapping $\theta : \mathbb{R} \rightarrow G$ with*

$$\theta(0) = e$$

and

$$\theta(t+d) = \theta(t) \cdot \varsigma(t)_d$$

for any $t \in \mathbb{R}$ and any $d \in D$.

From now on, we will assume the Lie group G to be regular, so that $\exp^G : \mathfrak{g} \rightarrow G$ is indeed a total function.

Notation 18 *Given $\xi \in \mathfrak{gl}(V)$ with ξ^{n+1} vanishing for some natural number n , we write*

$$e^\xi = \sum_{i=0}^n \frac{\xi^i}{i!}$$

It is easy to see that

Lemma 19 *Given $\xi \in \mathfrak{gl}(V)$ with ξ^{n+1} vanishing for some natural number n , we have*

$$\exp^{GL(V)} \xi = e^\xi$$

Proposition 20 *Given a homomorphism $\varphi : G \rightarrow H$ of Lie groups and $X \in \mathfrak{g}$, $\exp^H \varphi'(X)$ is defined, and we have*

$$\exp^H \varphi'(X) = \varphi(\exp^G X)$$

Remark 21 *The Lie group G is assumed to be regular, as we have said before, but the Lie group H is not assumed to be regular, so that \exp^H is not necessarily a total function.*

Proof. It suffices to note that, given a one-parameter subgroup $\theta : \mathbb{R} \rightarrow G$ of G with

$$\mathbf{d}\theta(i_D^{\mathbb{R}}) = X,$$

the mapping $\varphi \circ \theta : \mathbb{R} \rightarrow H$ is a one-parameter subgroup of H with

$$\mathbf{d}(\varphi \circ \theta)(i_D^{\mathbb{R}}) = \varphi'(X)$$

■

Proposition 22 *Given $X \in \mathfrak{g}$ with $(\text{ad } X)^{n+1}$ vanishing for some natural number n , we have*

$$\text{Ad}(\exp X) = e^{\text{ad } X}$$

Proof. We have

$$\begin{aligned} & \text{Ad}(\exp^G X) \\ &= \exp^{GL(V)}(\text{ad } X) \\ & \text{)By Proposition 20(} \\ &= e^{\text{ad } X} \\ & \text{)By Lemma 19(} \end{aligned}$$

■

We conclude this section by the following simple but significant proposition.

Proposition 23 *We have*

$$\exp t(dX) = X_{td}$$

for any $t \in \mathbb{R}$. In particular, we have

$$\exp dX = X_d$$

by setting

$$t = 1$$

Proof. For any $d' \in D$, we have

$$\begin{aligned} & (dX)_{t+d'} \\ &= X_{(t+d')d} \\ &= X_{td+d'd} \\ &= X_{td} \cdot X_{d'd} \\ & \text{)By Proposition 3(} \\ &= (dX)_t \cdot (dX)_{d'} \end{aligned}$$

so that we have the desired conclusion. ■

5 Logarithmic Derivatives

In this section we deal with the left and right derivations. First we deal with the left derivation.

Definition 24 *Given a microlinear space M and a function $f : M \rightarrow G$, the function*

$$\delta^{\text{left}} f : \mathbf{T}M \rightarrow \mathfrak{g}$$

is defined to be such that

$$(\mathbf{d}f(X))_d = f(x) \cdot (\delta^{\text{left}} f(X))_d$$

for any $x \in M$, any $X \in \mathbf{T}_x M$ and any $d \in D$. It is called the left logarithmic derivative of f . The restriction of $\delta^{\text{left}} f$ to $\mathbf{T}_x M$ is denoted by $\delta^{\text{left}} f(x)$.

The following is the Leibniz rule for the left logarithmic derivation.

Proposition 25 *Let M be a microlinear space. Given two functions*

$$f, g : M \rightarrow G$$

together with $X \in \mathbf{T}M$, we have

$$\delta^{\text{left}}(fg)(X) = \delta^{\text{left}} g(X) + \text{Ad} \left(g(x)^{-1} \right) (\delta^{\text{left}} f(X))$$

with

$$x = X_0$$

Proof. For any $d \in D$, we have

$$\begin{aligned} & (\delta^{\text{left}}(fg)(X))_d \\ &= g(x)^{-1} \cdot f(x)^{-1} \cdot f(X_d) \cdot g(X_d) \\ &= g(x)^{-1} \cdot f(x)^{-1} \cdot f(X_d) \cdot g(x) \cdot g(x)^{-1} \cdot g(X_d) \\ &= \left\{ \text{Ad} \left(g(x)^{-1} \right) (\delta^{\text{left}} f(X)) + \delta g(X) \right\}_d \\ & \text{)By Proposition 5(} \end{aligned}$$

so that we get the desired formula. ■

Theorem 26 *Given $X \in \mathfrak{g}$ with $(\text{ad } X)^{n+1}$ vanishing for some natural number n , we have*

$$\delta^{\text{left}}(\exp)(X) = \sum_{p=0}^n \frac{(-1)^p}{(p+1)!} (\text{ad } X)^p$$

Proof. The proof is essentially on the lines of Lemma 4.27 of [12]. We have

$$\begin{aligned}
& (s+t) \delta^{\text{left}}(\exp)((s+t)X) \\
&= \delta^{\text{left}}(\exp(s+t) \cdot)(X) \\
& \text{[By the chain rule of differentiation]} \\
&= \delta^{\text{left}}((\exp s \cdot)(\exp t \cdot))(X) \\
&= \delta^{\text{left}}(\exp t \cdot)(X) + \text{Ad}(\exp(-t)X)(\delta^{\text{left}}(\exp s \cdot))(X) \\
& \text{[By Proposition 25]} \\
&= t \delta^{\text{left}}(\exp)(tX) + \text{Ad}(\exp(-t)X)(s \delta^{\text{left}}(\exp)(sX))
\end{aligned}$$

so that, by letting

$$F(s) = s \delta^{\text{left}}(\exp)(sX)$$

so as to introduce a function

$$F : \mathbb{R} \rightarrow L(\mathfrak{g}, \mathfrak{g}),$$

we get

$$F(s+t) = F(t) + \text{Ad}(\exp(-t)X)(F(s)),$$

which earns us

$$F'(s) = F'(0) - (\text{ad } X)(F(s)) \quad (1)$$

by fixing s and differentiating with respect to t at $t = 0$. Since we have also

$$F'(s) = \delta^{\text{left}}(\exp)(sX) + s \delta^{\text{left}}(\exp)(X),$$

we get

$$F'(0) = \text{id}_{\mathfrak{g}},$$

by letting $s = 0$, so that the formula (1) is transmogrified into the ordinary differential equation

$$F'(s) = \text{id}_{\mathfrak{g}} - (\text{ad } X)(F(s))$$

on $L(\mathfrak{g}, \mathfrak{g})$. Its unique solution with the initial condition of $F(0)$'s vanishing is

$$F(s) = \sum_{p=0}^n \frac{(-1)^p s^{p+1}}{(p+1)!} (\text{ad } X)^p,$$

which results in the desired formula by letting $s = 1$. ■

Proposition 27 *Given $X, Y \in \mathfrak{g}$ with $[X, Y]$ vanishing, we have*

$$\exp X \cdot \exp Y = \exp X + Y$$

In particular, we have

$$\exp X \cdot \exp Y = \exp Y \cdot \exp X$$

Proof. Letting

$$H(t) = \exp X \cdot \exp tY \cdot \exp -(X + tY)$$

so as to get a function

$$H : \mathbb{R} \rightarrow G,$$

we have

$$H(0) = e$$

evidently. By differentiating H logarithmically, we have

$$\begin{aligned} \delta^{\text{left}} H(t) &= \delta^{\text{left}} (\exp) (- (X + tY)) (-Y) + \text{Ad} (\exp X + tY) (\delta^{\text{left}} (\exp) (tY) (Y)) \\ &= -Y + \text{Ad} (\exp X + tY) (Y) \\ &= -Y + e^{\text{ad} (X+tY)} (Y) \\ &= -Y + Y \\ &= 0 \end{aligned}$$

so that we have the desired formula. ■

Proposition 28 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2 \in D$, we have*

$$\begin{aligned} &\exp d_1 X \cdot \exp d_2 Y \\ &= \exp d_2 Y \cdot \exp d_1 X \cdot \exp d_1 d_2 [X, Y] \end{aligned}$$

Proof. we have

$$\begin{aligned} &\exp d_1 X + d_2 Y \\ &= \exp d_1 X \cdot \{ \delta^{\text{left}} (\exp) (d_1 X) (Y) \}_{d_2} \\ &\text{)logarithmic derivation(} \\ &= \exp d_1 X \cdot \left\{ Y - \frac{1}{2} d_1 [X, Y] \right\}_{d_2} \\ &\text{)By Theorem 26(} \\ &= \exp d_1 X \cdot Y_{d_2} \cdot \left(-\frac{1}{2} d_1 [X, Y] \right)_{d_2} \\ &\text{)By Proposition 5(} \\ &= \exp d_1 X \cdot \exp d_2 Y \cdot \exp -\frac{1}{2} d_1 d_2 [X, Y] \\ &\text{)By Proposition 23(,} \end{aligned}$$

while we have

$$\begin{aligned} &\exp d_1 X + d_2 Y \\ &= \exp d_2 Y + d_1 X \\ &= \exp d_2 Y \cdot \exp d_1 X \cdot \exp -\frac{1}{2} d_1 d_2 [Y, X] \end{aligned}$$

by the same token. Therefore we have

$$\begin{aligned} & \exp d_1 X \cdot \exp d_2 Y \cdot \exp -\frac{1}{2} d_1 d_2 [X, Y] \\ &= \exp d_2 Y \cdot \exp d_1 X \cdot \exp -\frac{1}{2} d_1 d_2 [Y, X] \end{aligned}$$

By multiplying

$$\exp \frac{1}{2} d_1 d_2 [X, Y]$$

from the right and making use of Proposition 27, we get the desired formula. ■

Now we deal with the right derivation.

Definition 29 *Given a microlinear space M and a function $f : M \rightarrow G$, the function*

$$\delta^{\text{right}} f : \mathbf{T}M \rightarrow \mathfrak{g}$$

is defined to be such that

$$(\mathbf{d}f(X))_d = (\delta^{\text{right}} f(X))_d \cdot f(x)$$

for any $x \in M$, any $X \in \mathbf{T}_x M$ and any $d \in D$. It is called the right logarithmic derivative of f . The restriction of $\delta^{\text{right}} f$ to $\mathbf{T}_x M$ is denoted by $\delta^{\text{right}} f(x)$.

Proposition 30 *Let M be a microlinear space. Given two functions*

$$f, g : M \rightarrow G$$

together with $X \in \mathbf{T}M$, we have

$$\delta^{\text{right}}(fg)(X) = \delta^{\text{right}} f(X) + \text{Ad}(f(x))(\delta^{\text{right}} g(X))$$

with

$$x = X_0$$

Theorem 31 *Given $X \in \mathfrak{g}$ with $(\text{ad } X)^{n+1}$ vanishing for some natural number n , we have*

$$\delta^{\text{right}}(\exp)(X) = \sum_{p=0}^n \frac{1}{(p+1)!} (\text{ad } X)^p$$

6 The Zassenhaus Formula

Lemma 32 *Given $d_1, \dots, d_n \in D$, we have*

$$\frac{(d_1 + \dots + d_n)^m}{m!} = \sum_{i_1 < \dots < i_m} d_{i_1} \dots d_{i_m}$$

for any natural number m with $m \leq n$.

Proof. The reader is referred to Lemma (p.10) of [9]. ■

Theorem 33 *Given $X, Y \in \mathfrak{g}$ and $d_1 \in D$, we have*

$$\begin{aligned} & \exp d_1 (X + Y) \\ &= \exp d_1 X. \exp d_1 Y \end{aligned}$$

Proof. We have

$$\begin{aligned} & \exp d_1 (X + Y) \\ &= (X + Y)_{d_1} \\ & \text{)By Proposition 23(} \\ &= X_{d_1}.Y_{d_1} \\ & \text{)By Proposition 5(} \\ &= \exp d_1 X. \exp d_1 Y \\ & \text{)By Proposition 23(} \end{aligned}$$

so that we have got to the desired formula. ■

Theorem 34 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2 \in D$, we have*

$$\begin{aligned} & \exp (d_1 + d_2) (X + Y) \\ &= \exp (d_1 + d_2) X. \exp (d_1 + d_2) Y. \exp -d_1 d_2 [X, Y] \\ &= \exp (d_1 + d_2) X. \exp (d_1 + d_2) Y. \exp -\frac{(d_1 + d_2)^2}{2} [X, Y] \end{aligned}$$

Proof. We have

$$\begin{aligned}
& \exp (d_1 + d_2) (X + Y) \\
&= \exp d_1 (X + Y) + d_2 (X + Y) \\
&= \exp d_1 (X + Y) \cdot \left\{ \delta^{\text{left}} (\exp) (d_1 (X + Y)) (X + Y) \right\}_{d_2} \\
& \text{)left logarithmic derivation(} \\
&= \exp d_1 (X + Y) \cdot (X + Y)_{d_2} \\
& \text{)By Theorem 26(} \\
&= \exp d_1 (X + Y) \cdot \exp d_2 (X + Y) \\
& \text{[By Proposition 23]} \\
&= \exp d_1 X \cdot \exp d_1 Y \cdot \exp d_2 X \cdot \exp d_2 Y \\
& \text{)By Theorem 33(} \\
&= \exp d_1 X \cdot \exp d_2 X \cdot \exp d_1 Y \cdot \exp d_1 d_2 [Y, X] \cdot \exp d_2 Y \\
& \text{)By Proposition 28(} \\
&= \exp d_1 X \cdot \exp d_2 X \cdot \exp d_1 Y \cdot \exp d_2 Y \cdot \exp d_1 d_2 [Y, X] \\
& \text{)By Proposition 27(} \\
&= \exp (d_1 + d_2) X \cdot \exp (d_1 + d_2) Y \cdot \exp d_1 d_2 [Y, X] \\
& \text{)By Proposition 27(} \\
&= \exp (d_1 + d_2) X \cdot \exp (d_1 + d_2) Y \cdot \exp -d_1 d_2 [X, Y]
\end{aligned}$$

so that we have got to the desired formula. ■

Theorem 35 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2, d_3 \in D$, we have*

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3) (X + Y) \\
&= \exp (d_1 + d_2 + d_3) X \cdot \exp (d_1 + d_2 + d_3) Y \cdot \exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] \cdot \\
& \exp d_1 d_2 d_3 [X + 2Y, [X, Y]] \\
&= \exp (d_1 + d_2 + d_3) X \cdot \exp (d_1 + d_2 + d_3) Y \cdot \exp - \frac{(d_1 + d_2 + d_3)^2}{2} [X, Y] \cdot \\
& \exp \frac{(d_1 + d_2 + d_3)^3}{12} [X + 2Y, [X, Y]]
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3) (X + Y) \\
&= \exp (d_1 + d_2) (X + Y) + d_3 (X + Y) \\
&= \exp (d_1 + d_2) (X + Y) \cdot \{\delta^{\text{left}}(\exp)((d_1 + d_2) (X + Y)) (X + Y)\}_{d_3} \\
&\text{)left logarithmic derivation(} \\
&= \exp (d_1 + d_2) (X + Y) \cdot (X + Y)_{d_3} \\
&\text{)By Theorem 26(} \\
&= \exp (d_1 + d_2) (X + Y) \cdot \exp d_3 (X + Y) \\
&\text{)By Proposition 23(} \\
&= \exp (d_1 + d_2) X \cdot \exp (d_1 + d_2) Y \cdot \exp -d_1 d_2 [X, Y] \cdot \exp d_3 X \cdot \exp d_3 Y \\
&\text{)By Theorems 33 and 34(} \\
&= \exp (d_1 + d_2) X \cdot \exp d_1 Y \cdot \exp d_2 Y \cdot \exp -d_1 d_2 [X, Y] \cdot \exp d_3 X \cdot \exp d_3 Y \\
&= \exp (d_1 + d_2 + d_3) X \cdot \exp d_1 Y \cdot \exp d_1 d_3 [Y, X] \cdot \exp d_2 Y \cdot \exp d_2 d_3 [Y, X] \cdot \\
&\exp -d_1 d_2 [X, Y] \cdot \exp -d_1 d_2 d_3 [[X, Y], X] \cdot \exp d_3 Y \\
&\left. \begin{aligned}
&\text{) By moving } \exp d_3 X \text{ left towards } \exp (d_1 + d_2) X \text{ via Propositions 27 and } \\
&\quad 28 \\
&= \exp (d_1 + d_2 + d_3) X \cdot \exp (d_1 + d_2) Y \cdot \exp d_1 d_3 [Y, X] \cdot \exp d_1 d_2 d_3 [[Y, X], Y] \cdot \\
&\exp d_2 d_3 [Y, X] \cdot \exp -d_1 d_2 [X, Y] \cdot \exp -d_1 d_2 d_3 [[X, Y], X] \cdot \exp d_3 Y \\
&\text{)By exchanging } \exp d_1 d_3 [Y, X] \text{ and } \exp d_2 Y \text{ via Proposition 28(} \\
&= \exp (d_1 + d_2 + d_3) X \cdot \exp (d_1 + d_2 + d_3) Y \cdot \exp d_1 d_3 [Y, X] \cdot \exp d_1 d_2 d_3 [[Y, X], Y] \cdot \\
&\exp d_2 d_3 [Y, X] \cdot \exp -d_1 d_2 [X, Y] \cdot \exp -d_1 d_2 d_3 [[X, Y], Y] \cdot \exp -d_1 d_2 d_3 [[X, Y], X] \\
&\left. \begin{aligned}
&\text{) By moving } \exp d_3 Y \text{ left towards } \exp (d_1 + d_2) Y \text{ via Propositions 27 and } \\
&\quad 28 \\
&= \exp (d_1 + d_2 + d_3) X \cdot \exp (d_1 + d_2 + d_3) Y \cdot \exp -(d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] \cdot \\
&\exp d_1 d_2 d_3 [X + 2Y, [X, Y]]
\end{aligned} \right)
\end{aligned}$$

so that we have got to the desired formula. ■

Theorem 36 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2, d_3, d_4 \in D$, we have*

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3 + d_4) (X + Y) \\
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y. \\
& \exp -(d_1 d_2 + d_1 d_3 + d_1 d_4 + d_2 d_3 + d_2 d_4 + d_3 d_4) [X, Y]. \\
& \exp (d_1 d_2 d_3 + d_1 d_2 d_4 + d_1 d_3 d_4 + d_2 d_3 d_4) [X + 2Y, [X, Y]]. \\
& \exp d_1 d_2 d_3 d_4 (-[X, [X, [X, Y]]] - 3[X, [Y, [X, Y]]] - 3[Y, [Y, [X, Y]]]) \\
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y. \\
& \exp -\frac{(d_1 + d_2 + d_3 + d_4)^2}{2} [X, Y]. \\
& \exp \frac{(d_1 + d_2 + d_3 + d_4)^3}{12} [X + 2Y, [X, Y]]. \\
& \exp \frac{(d_1 + d_2 + d_3 + d_4)^4}{24} (-[X, [X, [X, Y]]] - 3[X, [Y, [X, Y]]] - 3[Y, [Y, [X, Y]]])
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3 + d_4) (X + Y) \\
&= \exp (d_1 + d_2 + d_3) (X + Y) + d_4 (X + Y) \\
&= \exp (d_1 + d_2 + d_3) (X + Y) . \\
&\{ \delta^{\text{left}} (\exp ((d_1 + d_2 + d_3) (X + Y)) ((X + Y))) \}_{d_4} \\
&\text{)left logarithmic derivation(} \\
&= \exp (d_1 + d_2 + d_3) (X + Y) . (X + Y)_{d_4} \\
&\text{)By Theorem 26(} \\
&= \exp (d_1 + d_2 + d_3) (X + Y) . \exp d_4 (X + Y) \\
&\text{)By Proposition 23(} \\
&= \exp (d_1 + d_2 + d_3) X . \exp (d_1 + d_2 + d_3) Y . \exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] . \\
&\exp d_1 d_2 d_3 [X + 2Y, [X, Y]] . \exp d_4 X . \exp d_4 Y \\
&\text{)By Theorems 33 and 35(} \\
&= \exp (d_1 + d_2 + d_3 + d_4) X . \exp d_1 Y . \exp d_1 d_4 [Y, X] . \exp d_2 Y . \exp d_2 d_4 [Y, X] . \\
&\exp d_3 Y . \exp d_3 d_4 [Y, X] . \exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] . \\
&\exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) d_4 [[X, Y], X] . \exp d_1 d_2 d_3 [X + 2Y, [X, Y]] . \\
&\exp d_1 d_2 d_3 d_4 [[X + 2Y, [X, Y]], X] . \exp d_4 Y \\
&\left. \begin{array}{l} \text{) By moving } \exp d_4 X \text{ left towards } \exp (d_1 + d_2 + d_3) X \\ \text{via Propositions 27 and 28} \end{array} \right(\\
&= \exp (d_1 + d_2 + d_3 + d_4) X . \exp (d_1 + d_2) Y . \exp d_1 d_4 [Y, X] . \exp d_1 d_2 d_4 [[Y, X], Y] . \\
&\exp d_2 d_4 [Y, X] . \exp d_3 Y . \exp d_3 d_4 [Y, X] . \exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] . \\
&\exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) d_4 [[X, Y], X] . \exp d_1 d_2 d_3 [X + 2Y, [X, Y]] . \\
&\exp d_1 d_2 d_3 d_4 [[X + 2Y, [X, Y]], X] . \exp d_4 Y \\
&\text{)By interchanging } \exp d_1 d_4 [Y, X] \text{ and } \exp d_2 Y \text{ via Proposition 28(} \\
&= \exp (d_1 + d_2 + d_3 + d_4) X . \exp (d_1 + d_2 + d_3) Y . \exp d_1 d_4 [Y, X] . \exp d_1 d_3 d_4 [[Y, X], Y] . \\
&\exp d_1 d_2 d_4 [[Y, X], Y] . \exp d_1 d_2 d_3 d_4 [[[Y, X], Y], Y] . \exp d_2 d_4 [Y, X] . \\
&\exp d_2 d_3 d_4 [[Y, X], Y] . \exp d_3 d_4 [Y, X] . \exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] . \\
&\exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) d_4 [[X, Y], X] . \exp d_1 d_2 d_3 [X + 2Y, [X, Y]] . \\
&\exp d_1 d_2 d_3 d_4 [[X + 2Y, [X, Y]], X] . \exp d_4 Y \\
&\left. \begin{array}{l} \text{) By moving } \exp d_3 Y \text{ left towards } \exp (d_1 + d_2) Y \\ \text{via Propositions 27 and 28} \end{array} \right(
\end{aligned}$$

We keep on:

$$\begin{aligned}
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y. \exp d_1 d_4 [Y, X]. \\
&\exp d_1 d_3 d_4 [[Y, X], Y]. \exp d_1 d_2 d_4 [[Y, X], Y]. \exp d_1 d_2 d_3 d_4 [[[Y, X], Y], Y]. \\
&\exp d_2 d_4 [Y, X]. \exp d_2 d_3 d_4 [[Y, X], Y]. \exp d_3 d_4 [Y, X]. \\
&\exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y]. \exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) d_4 [[X, Y], Y]. \\
&\exp - (d_1 d_2 + d_1 d_3 + d_2 d_3) d_4 [[X, Y], X]. \exp d_1 d_2 d_3 [X + 2Y, [X, Y]]. \\
&\exp d_1 d_2 d_3 d_4 [[X + 2Y, [X, Y]], Y]. \exp d_1 d_2 d_3 d_4 [[X + 2Y, [X, Y]], X] \\
&\left. \vphantom{\exp} \right) \text{ By moving } \exp d_4 Y \text{ left towards } \exp (d_1 + d_2 + d_3) Y \left(\begin{array}{l} \text{via Propositions 27 and 28} \end{array} \right. \\
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y. \\
&\exp - (d_1 d_2 + d_1 d_3 + d_1 d_4 + d_2 d_3 + d_2 d_4 + d_3 d_4) [X, Y]. \\
&\exp (d_1 d_2 d_3 + d_1 d_2 d_4 + d_1 d_3 d_4 + d_2 d_3 d_4) [X + 2Y, [X, Y]]. \\
&\exp d_1 d_2 d_3 d_4 (-[X, [X, [X, Y]]] - 3[X, [Y, [X, Y]]] - 3[Y, [Y, [X, Y]]])
\end{aligned}$$

so that we have got to the desired formula. ■

We could keep on, but the complexity of computation increases rapidly.

7 The First Approach to the Baker-Campbell-Hausdorff Formula

The following result is no other than Theorem 33 itself.

Theorem 37 *Given $X, Y \in \mathfrak{g}$ and $d_1 \in D$, we have*

$$\begin{aligned}
&\exp d_1 X. \exp d_1 Y \\
&= \exp d_1 (X + Y)
\end{aligned}$$

Corollary 38 *Given $X_1, \dots, X_n \in \mathfrak{g}$ and $d_1 \in D$, we have*

$$\begin{aligned}
&\exp d_1 X_1. \exp d_1 X_2. \dots \exp d_1 X_n \\
&= \exp d_1 (X_1 + X_2 + \dots + X_n)
\end{aligned}$$

Proof. *By simple induction on n . ■*

Theorem 39 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2 \in D$, we have*

$$\begin{aligned}
&\exp (d_1 + d_2) X. \exp (d_1 + d_2) Y \\
&= \exp (d_1 + d_2) (X + Y) + d_1 d_2 [X, Y] \\
&= \exp (d_1 + d_2) (X + Y) + \frac{(d_1 + d_2)^2}{2} [X, Y]
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& \exp (d_1 + d_2) (X + Y) \\
&= \exp d_1 (X + Y) + d_2 (X + Y) \\
&= \exp d_1 (X + Y) \cdot \left\{ \delta^{\text{left}} (\exp) (d_1 (X + Y)) (X + Y) \right\}_{d_2} \\
& \text{)left logarithmic derivation(} \\
&= \exp d_1 (X + Y) \cdot (X + Y)_{d_2} \\
& \text{)By Theorem 26(} \\
&= \exp d_1 (X + Y) \cdot \exp d_2 (X + Y) \\
& \text{)By Proposition 23(} \\
&= \exp d_1 X \cdot \exp d_1 Y \cdot \exp d_2 X \cdot \exp d_2 Y \\
& \text{)By Theorem 33(} \\
&= \exp d_1 X \cdot \exp d_2 X \cdot \exp d_1 Y \cdot \exp d_1 d_2 [Y, X] \cdot \exp d_2 Y \\
& \text{)By Proposition 28(} \\
&= \exp d_1 X \cdot \exp d_2 X \cdot \exp d_1 Y \cdot \exp d_2 Y \cdot \exp d_1 d_2 [Y, X] \\
& \text{)By Proposition 27(} \\
&= \exp (d_1 + d_2) X \cdot \exp (d_1 + d_2) Y \cdot \exp d_1 d_2 [Y, X] \\
& \text{)By Proposition 27(}
\end{aligned}$$

so that we get the desired formula by multiplying

$$\exp d_1 d_2 [X, Y]$$

from the right and making use of Proposition 27. ■

Corollary 40 (*cf. Theorem 2.12.4 of [19]*). Given $X_1, \dots, X_n \in \mathfrak{g}$ and $d_1, d_2 \in D$, we have

$$\begin{aligned}
& \exp (d_1 + d_2) X_1 \cdot \exp (d_1 + d_2) X_2 \dots \exp (d_1 + d_2) X_n \\
&= \exp (d_1 + d_2) (X_1 + \dots + X_n) + d_1 d_2 \sum_{1 \leq i < j \leq n} [X_i, X_j] \\
&= \exp (d_1 + d_2) (X_1 + \dots + X_n) + \frac{(d_1 + d_2)^2}{2} \sum_{1 \leq i < j \leq n} [X_i, X_j]
\end{aligned}$$

Proof. Here we deal only with the case of $n = 3$, leaving the general treatment by induction on n to the reader. We note in passing that the case of

$n = 2$ is no other than Theorem 39 itself. We have

$$\begin{aligned}
& \exp (d_1 + d_2) X_1 . \exp (d_1 + d_2) X_2 . \exp (d_1 + d_2) X_3 \\
&= \exp (d_1 + d_2) (X_1 + X_2) + \frac{(d_1 + d_2)^2}{2} [X_1, X_2] . \exp (d_1 + d_2) X_3 \\
& \text{)By Theorem 39(} \\
&= \exp (d_1 + d_2) \left\{ (X_1 + X_2) + \frac{d_1 + d_2}{2} [X_1, X_2] \right\} . \exp (d_1 + d_2) X_3 \\
&= \exp (d_1 + d_2) \left\{ (X_1 + X_2 + X_3) + \frac{d_1 + d_2}{2} [X_1, X_2] \right\} + \\
& d_1 d_2 \left[(X_1 + X_2) + \frac{d_1 + d_2}{2} [X_1, X_2], X_3 \right] \\
& \text{)By Theorem 39(} \\
&= \exp (d_1 + d_2) (X_1 + X_2 + X_3) + d_1 d_2 ([X_1, X_2] + [X_1, X_3] + [X_2, X_3])
\end{aligned}$$

so that we are done. ■

Theorem 41 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2, d_3 \in D$, we have*

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3) X . \exp (d_1 + d_2 + d_3) Y \\
&= \exp (d_1 + d_2 + d_3) (X + Y) + (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] + \\
& \frac{1}{2} d_1 d_2 d_3 [X - Y, [X, Y]] \\
&= \exp (d_1 + d_2 + d_3) X + (d_1 + d_2 + d_3) Y + \frac{(d_1 + d_2 + d_3)^2}{2} [X, Y] + \\
& \frac{(d_1 + d_2 + d_3)^3}{12} [X - Y, [X, Y]]
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3) (X + Y) + (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] \\
&= \exp \{ (d_1 + d_2) (X + Y) + d_1 d_2 [X, Y] \} + d_3 \{ (X + Y) + (d_1 + d_2) [X, Y] \} \\
&= \exp ((d_1 + d_2) (X + Y) + d_1 d_2 [X, Y]) . \\
&\{ \delta^{\text{left}} (\exp ((d_1 + d_2) (X + Y) + d_1 d_2 [X, Y]) ((X + Y) + (d_1 + d_2) [X, Y])) \}_{d_3} \\
&\text{)left logarithmic derivation(} \\
&= \exp ((d_1 + d_2) (X + Y) + d_1 d_2 [X, Y]) . \\
&\left\{ (X + Y) + (d_1 + d_2) [X, Y] - \frac{1}{2} d_1 d_2 [X + Y, [X, Y]] \right\}_{d_3} \\
&\text{)By Theorem 26(} \\
&= \exp (d_1 + d_2) X . \exp (d_1 + d_2) Y . \\
&\left\{ (X + Y) + (d_1 + d_2) [X, Y] - \frac{1}{2} d_1 d_2 [X + Y, [X, Y]] \right\}_{d_3} \\
&\text{[By Theorem 39]} \\
&= \exp (d_1 + d_2) X . \exp (d_1 + d_2) Y . (X + Y)_{d_3} . ((d_1 + d_2) [X, Y])_{d_3} . \\
&\left(-\frac{1}{2} d_1 d_2 [X + Y, [X, Y]] \right)_{d_3} \\
&\text{[By Proposition 5]} \\
&= \exp (d_1 + d_2) X . \exp (d_1 + d_2) Y . \exp d_3 (X + Y) . \exp (d_1 + d_2) d_3 [X, Y] . \\
&\exp -\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, Y]] \\
&\text{[By Proposition 23]} \\
&= \exp (d_1 + d_2) X . \exp d_1 Y . \exp d_2 Y . \exp d_3 X . \exp d_3 Y . \exp (d_1 + d_2) d_3 [X, Y] . \\
&\exp -\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, Y]] \\
&\text{)By Proposition 27(} \\
&= \exp (d_1 + d_2) X . \exp d_1 Y . \exp d_3 X . \exp d_2 Y . \exp d_2 d_3 [Y, X] . \exp d_3 Y . \\
&\exp (d_1 + d_2) d_3 [X, Y] . \exp -\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, Y]] \\
&\text{)By Proposition 28(} \\
&= \exp (d_1 + d_2) X . \exp d_3 X . \exp d_1 Y . \exp d_1 d_3 [Y, X] . \exp d_2 Y . \exp d_2 d_3 [Y, X] . \exp d_3 Y . \\
&\exp (d_1 + d_2) d_3 [X, Y] . \exp -\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, Y]] \\
&\text{)By Proposition 28(}
\end{aligned}$$

We keep on.

$$= \exp (d_1 + d_2) X. \exp d_3 X. \exp d_1 Y. \exp d_2 Y. \exp d_1 d_3 [Y, X]. \exp d_1 d_2 d_3 [[Y, X], Y]. \\ \exp d_2 d_3 [Y, X]. \exp d_3 Y. \exp (d_1 + d_2) d_3 [X, Y]. \exp -\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, Y]]$$

)By Proposition 28(

$$= \exp (d_1 + d_2 + d_3) X. \exp (d_1 + d_2 + d_3) Y. \exp \frac{1}{2} d_1 d_2 d_3 [Y - X, [X, Y]]$$

)By repeated use of Proposition 27(

so that we get the desired formula by multiplying

$$\exp \frac{1}{2} d_1 d_2 d_3 [X - Y, [X, Y]]$$

from the right and making use of Proposition 27. ■

Theorem 42 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2, d_3, d_4 \in D$, we have*

$$\begin{aligned} & \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y \\ &= \exp (d_1 + d_2 + d_3 + d_4) X + (d_1 + d_2 + d_3 + d_4) Y + \\ & (d_1 d_2 + d_1 d_3 + d_1 d_4 + d_2 d_3 + d_2 d_4 + d_3 d_4) [X, Y] + \\ & \frac{1}{2} (d_1 d_2 d_3 + d_1 d_2 d_4 + d_1 d_3 d_4 + d_2 d_3 d_4) [X - Y, [X, Y]] - \\ & d_1 d_2 d_3 d_4 \left(\frac{1}{2} [X, [X, [X, Y]]] + \frac{1}{2} [Y, [Y, [X, Y]]] + 2 [X, [Y, [X, Y]]] \right) \\ &= \exp (d_1 + d_2 + d_3 + d_4) X + (d_1 + d_2 + d_3 + d_4) Y + \\ & \frac{(d_1 + d_2 + d_3 + d_4)^2}{2} [X, Y] + \\ & \frac{(d_1 + d_2 + d_3 + d_4)^3}{12} [X - Y, [X, Y]] - \\ & \frac{(d_1 + d_2 + d_3 + d_4)^4}{24} \left(\frac{1}{2} [X, [X, [X, Y]]] + \frac{1}{2} [Y, [Y, [X, Y]]] + 2 [X, [Y, [X, Y]]] \right) \end{aligned}$$

Proof. We have

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3 + d_4) X + (d_1 + d_2 + d_3 + d_4) Y + \\
& (d_1 d_2 + d_1 d_3 + d_1 d_4 + d_2 d_3 + d_2 d_4 + d_3 d_4) [X, Y] + \\
& \frac{1}{2} (d_1 d_2 d_3 + d_1 d_2 d_4 + d_1 d_3 d_4 + d_2 d_3 d_4) [X - Y, [X, Y]] \\
& = \exp \left\{ \begin{array}{c} (d_1 + d_2 + d_3) (X + Y) + (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] + \\ \frac{1}{2} d_1 d_2 d_3 [X - Y, [X, Y]] \end{array} \right\} + \\
& \left\{ d_4 (X + Y) + d_4 (d_1 + d_2 + d_3) [X, Y] + \frac{1}{2} d_4 (d_1 d_2 + d_1 d_3 + d_2 d_3) [X - Y, [X, Y]] \right\} \\
& = \exp \left\{ \begin{array}{c} (d_1 + d_2 + d_3) X + (d_1 + d_2 + d_3) Y + (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] + \\ \frac{1}{2} d_1 d_2 d_3 [X - Y, [X, Y]] \end{array} \right\} \cdot \\
& \left\{ \delta^{\text{left}} (\exp) \left(\begin{array}{c} (d_1 + d_2 + d_3) (X + Y) + (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] + \\ \frac{1}{2} d_1 d_2 d_3 [X - Y, [X, Y]] \end{array} \right) \right. \\
& \left. \left(\begin{array}{c} (X + Y) + (d_1 + d_2 + d_3) [X, Y] + \\ \frac{1}{2} (d_1 d_2 + d_1 d_3 + d_2 d_3) [X - Y, [X, Y]] \end{array} \right) \right\}_{d_4} \\
&) \text{left logarithmic derivation} (\\
& = \exp \left\{ \begin{array}{c} (d_1 + d_2 + d_3) (X + Y) + (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] + \\ \frac{1}{2} d_1 d_2 d_3 [X - Y, [X, Y]] \end{array} \right\} \cdot \\
& \left\{ \begin{array}{c} (X + Y) + (d_1 + d_2 + d_3) [X, Y] + \frac{1}{2} (d_1 d_2 + d_1 d_3 + d_2 d_3) [X - Y, [X, Y]] - \\ \frac{1}{2} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X - Y, [X, Y]]] + \\ \frac{1}{2} (d_1 d_2 + d_1 d_3 + d_2 d_3) [X + Y, [X, Y]] + \frac{1}{4} d_1 d_2 d_3 [X + Y, [X - Y, [X, Y]]] + \\ \frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{array} \right\}_{d_4} \\
&) \text{By Theorem 26} (\\
& = \exp \left\{ \begin{array}{c} (d_1 + d_2 + d_3) (X + Y) + (d_1 d_2 + d_1 d_3 + d_2 d_3) [X, Y] + \\ \frac{1}{2} d_1 d_2 d_3 [X - Y, [X, Y]] \end{array} \right\} \cdot \\
& \left\{ \begin{array}{c} (X + Y) + (d_1 + d_2 + d_3) [X, Y] - (d_1 d_2 + d_1 d_3 + d_2 d_3) [Y, [X, Y]] + \\ \frac{1}{2} d_1 d_2 d_3 [X + Y, [X, [X, Y]]] + \frac{3}{2} d_1 d_2 d_3 [X + Y, [Y, [X, Y]]] \end{array} \right\}_{d_4} \\
& = \exp (d_1 + d_2 + d_3) X \cdot \exp (d_1 + d_2 + d_3) Y \cdot \\
& \left\{ \begin{array}{c} (X + Y) + (d_1 + d_2 + d_3) [X, Y] - (d_1 d_2 + d_1 d_3 + d_2 d_3) [Y, [X, Y]] + \\ \frac{1}{2} d_1 d_2 d_3 [X + Y, [X, [X, Y]]] + \frac{3}{2} d_1 d_2 d_3 [X + Y, [Y, [X, Y]]] \end{array} \right\}_{d_4} \\
&) \text{By Theorem 41} (\\
& = \exp (d_1 + d_2 + d_3) X \cdot \exp (d_1 + d_2 + d_3) Y \cdot (X + Y)_{d_4} \cdot ((d_1 + d_2 + d_3) [X, Y])_{d_4} \cdot \\
& (- (d_1 d_2 + d_1 d_3 + d_2 d_3) [Y, [X, Y]])_{d_4} \cdot \\
& \left(\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, [X, Y]]] + \frac{3}{2} d_1 d_2 d_3 [X + Y, [Y, [X, Y]]] \right)_{d_4} \\
&) \text{By Proposition 5} (\\
& = \exp (d_1 + d_2 + d_3) X \cdot \exp (d_1 + d_2 + d_3) Y \cdot \exp_{d_4} (X + Y) \cdot \exp_{d_4} (d_1 + d_2 + d_3) [X, Y] \cdot \\
& \exp_{d_4} (-d_4 (d_1 d_2 + d_1 d_3 + d_2 d_3) [Y, [X, Y]]) \cdot \\
& \exp_{d_4} \left(\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, [X, Y]]] + \frac{3}{2} d_1 d_2 d_3 [X + Y, [Y, [X, Y]]] \right) \\
&) \text{By Proposition 23} (\\
& = \exp (d_1 + d_2 + d_3) X \cdot \exp_{d_1} Y \cdot \exp_{d_2} Y \cdot \exp_{d_3} Y \cdot \exp_{d_4} X \cdot \exp_{d_4} Y \cdot \\
& \exp_{d_4} (d_1 + d_2 + d_3) [X, Y] \cdot \exp_{d_4} (-d_4 (d_1 d_2 + d_1 d_3 + d_2 d_3) [Y, [X, Y]]) \cdot \\
& \exp_{d_4} \left(\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, [X, Y]]] + \frac{3}{2} d_1 d_2 d_3 [X + Y, [Y, [X, Y]]] \right) \\
&) \text{By Proposition 27} (
\end{aligned}$$

We keep on

$$\begin{aligned}
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp d_1 Y. \exp d_1 d_4 [Y, X]. \exp d_2 Y. \exp d_2 d_4 [Y, X]. \exp d_3 Y. \\
&\exp d_3 d_4 [Y, X]. \exp d_4 Y. \exp d_4 (d_1 + d_2 + d_3) [X, Y]. \\
&\exp -d_4 (d_1 d_2 + d_1 d_3 + d_2 d_3) [Y, [X, Y]]. \\
&\exp d_4 \left(\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, [X, Y]]] + \frac{3}{2} d_1 d_2 d_3 [X + Y, [Y, [X, Y]]] \right) \\
&\left. \begin{array}{l} \text{By moving } \exp d_4 X \text{ left towards } \exp (d_1 + d_2 + d_3) X \\ \text{via Propositions 27 and 28} \end{array} \right(\\
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2) Y. \exp d_1 d_4 [Y, X]. \exp d_1 d_2 d_4 [[Y, X], Y]. \\
&\exp d_2 d_4 [Y, X]. \exp d_3 Y. \exp d_3 d_4 [Y, X]. \exp d_4 Y. \exp d_4 (d_1 + d_2 + d_3) [X, Y]. \\
&\exp -d_4 (d_1 d_2 + d_1 d_3 + d_2 d_3) [Y, [X, Y]]. \\
&\exp d_4 \left(\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, [X, Y]]] + \frac{3}{2} d_1 d_2 d_3 [X + Y, [Y, [X, Y]]] \right) \\
&\left. \begin{array}{l} \text{By exchanging } \exp d_1 d_4 [Y, X] \text{ and } \exp d_2 Y \text{ via Proposition 28} \\ \text{and using Proposition 27} \end{array} \right(\\
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3) Y. \exp d_1 d_4 [Y, X]. \exp d_1 d_3 d_4 [[Y, X], Y]. \\
&\exp d_1 d_2 d_4 [[Y, X], Y]. \exp d_1 d_2 d_3 d_4 [[[Y, X], Y], Y]. \exp d_2 d_4 [Y, X]. \\
&\exp d_2 d_3 d_4 [[Y, X], Y]. \exp d_3 d_4 [Y, X]. \exp d_4 Y. \exp d_4 (d_1 + d_2 + d_3) [X, Y]. \\
&\exp -d_4 (d_1 d_2 + d_1 d_3 + d_2 d_3) [Y, [X, Y]]. \\
&\exp d_4 \left(\frac{1}{2} d_1 d_2 d_3 [X + Y, [X, [X, Y]]] + \frac{3}{2} d_1 d_2 d_3 [X + Y, [Y, [X, Y]]] \right) \\
&\left. \begin{array}{l} \text{By moving } \exp d_3 Y \text{ left towards } \exp (d_1 + d_2) Y \\ \text{via Propositions 27 and 28} \end{array} \right(\\
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y. \\
&\exp d_1 d_2 d_3 d_4 \left([[[[Y, X], Y], Y] + \frac{1}{2} [X + Y, [X, [X, Y]]] + \frac{3}{2} [X + Y, [Y, [X, Y]]] \right) \\
&\left. \begin{array}{l} \text{By moving } \exp d_4 Y \text{ left towards } \exp (d_1 + d_2 + d_3) Y \text{ via Proposition 27} \end{array} \right(\\
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y. \\
&\exp d_1 d_2 d_3 d_4 \left(\frac{1}{2} [X, [X, [X, Y]]] + \frac{1}{2} [Y, [X, [X, Y]]] + \frac{3}{2} [X, [Y, [X, Y]]] + \frac{1}{2} [Y, [Y, [X, Y]]] \right) \\
&= \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y. \\
&\exp d_1 d_2 d_3 d_4 \left(\frac{1}{2} [X, [X, [X, Y]]] + \frac{1}{2} [Y, [Y, [X, Y]]] + 2 [X, [Y, [X, Y]]] \right) \\
&\left. \begin{array}{l} \text{By Lemma 10} \end{array} \right(
\end{aligned}$$

so that we get the desired formula by multiplying

$$\exp -d_1 d_2 d_3 d_4 \left(\frac{1}{2} [X, [X, [X, Y]]] + \frac{1}{2} [Y, [Y, [X, Y]]] + 2 [X, [Y, [X, Y]]] \right)$$

from the right and making use of Proposition 27. ■

We could keep on, but the complexity of computation increases rapidly.

8 The Second Approach to the Baker-Campbell-Hausdorff Formula

Theorem 43 *Given $X, Y \in \mathfrak{g}$ and $d_1 \in D$, we have*

$$\begin{aligned} & \exp d_1 X . \exp d_1 Y \\ &= \exp d_1 (X + Y) \end{aligned}$$

Proof. By Proposition 27. ■

Theorem 44 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2 \in D$, we have*

$$\begin{aligned} & \exp (d_1 + d_2) X . \exp (d_1 + d_2) Y \\ &= \exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] \end{aligned}$$

Proof. We have

$$\begin{aligned} & \exp (d_1 + d_2) X . \exp (d_1 + d_2) Y \\ &= \exp d_1 X + d_2 X . \exp d_1 Y + d_2 Y \\ &= \exp d_2 X . \exp d_1 X . \exp d_1 Y . \exp d_2 Y \\ & \text{)By Proposition 27(} \\ &= \exp d_2 X . \exp d_1 (X + Y) . \exp d_2 Y \\ & \text{)By Theorem 43(} \\ &= \exp d_2 X . \exp d_1 (X + Y) . \exp d_2 \left\{ Y - \frac{1}{2} d_1 [X, Y] \right\} . \exp \frac{1}{2} d_1 d_2 [X, Y] \\ &= \exp d_2 X . \exp d_1 (X + Y) + d_2 Y . \exp \frac{1}{2} d_1 d_2 [X, Y] \\ & \text{)By Theorem 26 with } \delta^{\text{left}}(\exp)(d_1(X + Y))(Y) = Y - \frac{1}{2} d_1 [X, Y] \text{(} \\ &= \exp -\frac{1}{2} d_1 d_2 [Y, X] . \exp d_2 \left\{ X + \frac{1}{2} d_1 [Y, X] \right\} . \exp d_1 (X + Y) + d_2 Y . \exp \frac{1}{2} d_1 d_2 [X, Y] \\ &= \exp -\frac{1}{2} d_1 d_2 [Y, X] . \exp (d_1 + d_2) (X + Y) . \exp \frac{1}{2} d_1 d_2 [X, Y] \\ & \text{)By Theorem 31 with } \delta^{\text{right}}(\exp)(d_1(X + Y) + d_2 X)(X) = X + \frac{1}{2} d_1 [Y, X] \text{(} \\ &= \exp (d_1 + d_2) (X + Y) + d_1 d_2 [X, Y] \end{aligned}$$

■

Theorem 45 Given $X, Y \in \mathfrak{g}$ and $d_1, d_2, d_3 \in D$, we have

$$\begin{aligned} & \exp (d_1 + d_2 + d_3) X . \exp (d_1 + d_2 + d_3) Y \\ &= \exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ & \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] \end{aligned}$$

Proof. We have

$$\begin{aligned} & \exp (d_1 + d_2 + d_3) X . \exp (d_1 + d_2 + d_3) Y \\ &= \exp (d_1 + d_2) X + d_3 X . \exp (d_1 + d_2) Y + d_3 Y \\ &= \exp d_3 X . \exp (d_1 + d_2) X . \exp (d_1 + d_2) Y . \exp d_3 Y \\ & \text{)By Proposition 27(} \\ &= \exp d_3 X . \exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] . \exp d_3 Y \\ & \text{)By Theorem 44(} \\ &= \exp d_3 X . \exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] . \\ & \exp d_3 \left\{ \begin{array}{l} Y - \frac{1}{2} \left((d_1 + d_2) [X, Y] + \frac{1}{2} (d_1 + d_2)^2 [[X, Y], Y] \right) + \\ \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] \end{array} \right\} . \\ & \exp d_3 \left\{ \begin{array}{l} \frac{1}{2} \left((d_1 + d_2) [X, Y] + \frac{1}{2} (d_1 + d_2)^2 [[X, Y], Y] \right) - \\ \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] \end{array} \right\} \\ &= \exp d_3 X . \exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] + d_3 Y . \\ & \exp d_3 \left\{ \begin{array}{l} \frac{1}{2} \left((d_1 + d_2) [X, Y] + \frac{1}{2} (d_1 + d_2)^2 [[X, Y], Y] \right) - \\ \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] \end{array} \right\} \\ & \left. \begin{array}{l} \text{By Theorem 26} \\ \delta^{\text{left}}(\exp) \left((d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] \right) (Y) \\ \text{with } = Y - \frac{1}{2} \left((d_1 + d_2) [X, Y] + \frac{1}{2} (d_1 + d_2)^2 [[X, Y], Y] \right) + \\ \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] \end{array} \right) \left(\right. \end{aligned}$$

We keep on.

$$\begin{aligned}
&= \exp d_3 X. \exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] + d_3 Y. \\
&\exp d_3 \left\{ \frac{1}{2} (d_1 + d_2) [X, Y] - \frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]] \right\}. \\
&\exp d_3 \left\{ \begin{array}{c} \frac{1}{4} (d_1 + d_2)^2 [[X, Y], Y] - \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] + \\ \frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]] \end{array} \right\} \\
&= \exp d_3 X. \\
&\exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] + d_3 \left\{ Y + \frac{1}{2} (d_1 + d_2) [X, Y] \right\}. \\
&\exp d_3 \left\{ \begin{array}{c} \frac{1}{4} (d_1 + d_2)^2 [[X, Y], Y] - \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] + \\ \frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]] \end{array} \right\} \\
&\left. \begin{array}{c} \text{By Theorem 26 with} \\ \delta^{\text{left}} (\exp) \left((d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] + d_3 Y \right) \\ \left(\begin{array}{c} \frac{1}{2} (d_1 + d_2) [X, Y] \\ \frac{1}{2} (d_1 + d_2) [X, Y] - \frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]] \end{array} \right) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
&= \exp d_3 \left\{ -\frac{1}{2} \left((d_1 + d_2) [Y, X] + \frac{1}{2} (d_1 + d_2)^2 [[X, Y], X] \right) - \frac{\frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]]}{\frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]]} \right\}. \\
&\exp d_3 \left\{ X + \frac{1}{2} \left((d_1 + d_2) [Y, X] + \frac{1}{2} (d_1 + d_2)^2 [[X, Y], X] \right) + \frac{\frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]]}{\frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]]} \right\}. \\
&\exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] + d_3 \left\{ Y + \frac{1}{2} (d_1 + d_2) [X, Y] \right\}. \\
&\exp d_3 \left\{ \frac{1}{4} (d_1 + d_2)^2 [[X, Y], Y] - \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] + \frac{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]]}{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]]} \right\} \\
&= \exp d_3 \left\{ -\frac{1}{2} \left((d_1 + d_2) [Y, X] + \frac{1}{2} (d_1 + d_2)^2 [[X, Y], X] \right) - \frac{\frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]]}{\frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]]} \right\}. \\
&\exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] + d_3 \left\{ (X + Y) + \frac{1}{2} (d_1 + d_2) [X, Y] \right\}. \\
&\exp d_3 \left\{ \frac{1}{4} (d_1 + d_2)^2 [[X, Y], Y] - \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] + \frac{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]]}{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]]} \right\} \\
&\left(\begin{array}{c} \text{By Theorem 31 with} \\ \delta^{\text{right}}(\exp) \left(\begin{array}{c} (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] + \\ d_3 \left\{ Y + \frac{1}{2} (d_1 + d_2) [X, Y] \right\} \end{array} \right) (X) \\ = X + \frac{1}{2} \left((d_1 + d_2) [Y, X] + \frac{1}{2} (d_1 + d_2)^2 [[X, Y], X] \right) + \\ \frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]] \end{array} \right) \left(\begin{array}{c} \\ \\ \\ \end{array} \right) \\
&= \exp d_3 \left\{ -\frac{1}{4} (d_1 + d_2)^2 [[X, Y], X] - \frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]] + \frac{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [Y, X]]}{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [Y, X]]} \right\}. \\
&\exp d_3 \left\{ -\frac{1}{2} (d_1 + d_2) [Y, X] - \frac{1}{4} (d_1 + d_2)^2 [X + Y, [Y, X]] \right\}. \\
&\exp (d_1 + d_2) (X + Y) + \frac{1}{2} (d_1 + d_2)^2 [X, Y] + d_3 \left\{ (X + Y) + \frac{1}{2} (d_1 + d_2) [X, Y] \right\}. \\
&\exp d_3 \left\{ \frac{1}{4} (d_1 + d_2)^2 [[X, Y], Y] - \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] + \frac{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]]}{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]]} \right\} \\
&= \exp d_3 \left\{ -\frac{1}{4} (d_1 + d_2)^2 [[X, Y], X] - \frac{1}{6} (d_1 + d_2)^2 [X + Y, [Y, X]] + \frac{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [Y, X]]}{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [Y, X]]} \right\}. \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y]. \\
&\exp d_3 \left\{ \frac{1}{4} (d_1 + d_2)^2 [[X, Y], Y] - \frac{1}{6} (d_1 + d_2)^2 [X + Y, [X, Y]] + \frac{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]]}{\frac{1}{4} (d_1 + d_2)^2 [X + Y, [X, Y]]} \right\} \\
&\left(\begin{array}{c} \text{By Theorem 31 with} \\ \delta^{\text{right}}(\exp) \left(\begin{array}{c} (d_1 + d_2) (X + Y) + \\ \frac{1}{2} (d_1 + d_2)^2 [X, Y] + \\ d_3 \left\{ (X + Y) + \frac{1}{2} (d_1 + d_2) [X, Y] \right\} \end{array} \right) \left(-\frac{1}{2} (d_1 + d_2) [Y, X] \right) \\ = -\frac{1}{2} (d_1 + d_2) [Y, X] - \frac{1}{24} (d_1 + d_2)^2 [X + Y, [Y, X]] \end{array} \right) \left(\begin{array}{c} \\ \\ \\ \end{array} \right) \\
&= \exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{4} (d_1 + d_2)^2 d_3 ([X, [X, Y]] - [Y, [X, Y]]) \\
&= \exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]]
\end{aligned}$$

■

Theorem 46 *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2, d_3, d_4 \in D$, we have*

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3 + d_4) X . \exp (d_1 + d_2 + d_3 + d_4) Y \\
&= \exp (d_1 + d_2 + d_3 + d_4) (X + Y) + \frac{(d_1 + d_2 + d_3 + d_4)^2}{2} [X, Y] + \\
& \frac{(d_1 + d_2 + d_3 + d_4)^3}{12} [X - Y, [X, Y]] - \\
& \frac{(d_1 + d_2 + d_3 + d_4)^4}{48} \left(\begin{array}{c} [X, [Y, [X, Y]] + [Y, [X, [X, Y]] + \\ [X + Y, [X + Y, [X, Y]] \end{array} \right)
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3 + d_4) X. \exp (d_1 + d_2 + d_3 + d_4) Y \\
&= \exp (d_1 + d_2 + d_3) X + d_4 X. \exp (d_1 + d_2 + d_3) Y + d_4 Y \\
&= \exp d_4 X. \exp (d_1 + d_2 + d_3) X. \exp (d_1 + d_2 + d_3) Y. \exp d_4 Y \\
& \text{)By Proposition 27(} \\
&= \exp d_4 X. \\
& \exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
& \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] . \\
& \exp d_4 Y \\
& \text{)By Theorem 45(} \\
&= \exp d_4 X. \\
& \exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
& \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] . \\
& \exp d_4 \left\{ \begin{array}{l} Y - \frac{1}{2} \left(\begin{array}{l} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{2} (d_1 + d_2 + d_3)^2 [[X, Y], Y] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] \end{array} \right) + \\ \frac{1}{6} \left(\begin{array}{l} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ \frac{1}{2} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] \end{array} \right) \end{array} \right\} . \\
& \exp d_4 \left\{ \begin{array}{l} \frac{1}{2} \left(\begin{array}{l} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{2} (d_1 + d_2 + d_3)^2 [[X, Y], Y] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] \end{array} \right) - \\ \frac{1}{6} \left(\begin{array}{l} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ \frac{1}{2} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] \end{array} \right) \end{array} \right\} \\
&= \exp d_4 X. \\
& \exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
& \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + d_4 Y. \\
& \exp d_4 \left\{ \begin{array}{l} \frac{1}{2} \left(\begin{array}{l} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{2} (d_1 + d_2 + d_3)^2 [[X, Y], Y] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] \end{array} \right) - \\ \frac{1}{6} \left(\begin{array}{l} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ \frac{1}{2} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] \end{array} \right) \end{array} \right\} \\
& \left. \begin{array}{l} \text{By Theorem 26 with} \\ \delta^{\text{left}} (\exp) \left(\begin{array}{l} (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] \end{array} \right) (Y) \\ = Y - \frac{1}{2} \left(\begin{array}{l} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{2} (d_1 + d_2 + d_3)^2 [[X, Y], Y] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] \end{array} \right) + \\ \frac{1}{6} \left(\begin{array}{l} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ \frac{1}{2} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] \end{array} \right) \end{array} \right)
\end{aligned}$$

We keep on.

$$\begin{aligned}
&= \exp d_4 X. \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + d_4 Y. \\
&\exp d_4 \left\{ \begin{aligned} &\frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \\ &\frac{1}{2} \left(\begin{aligned} &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right) + \end{aligned} \right\}. \\
&\exp d_4 \left\{ \begin{aligned} &\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ &\frac{1}{2} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right\} - \\
&= \exp d_4 X. \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ \begin{aligned} &Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \end{aligned} \right\}. \\
&\exp d_4 \left\{ \begin{aligned} &\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ &\frac{1}{2} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right\} - \\
&\left(\begin{aligned} &\text{By Theorem 26 with} \\ &\delta^{\text{left}} (\exp) \left(\begin{aligned} &(d_1 + d_2 + d_3) (X + Y) + \\ &\frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\ &d_4 Y \end{aligned} \right) \\ &\left(\begin{aligned} &\frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \end{aligned} \right) \\ &= \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \\ &\frac{1}{2} \left(\begin{aligned} &\frac{1}{2} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right) + \end{aligned} \right)
\end{aligned}$$

We keep on again.

$$\begin{aligned}
&= \exp d_4 X. \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \right. \\
&\quad \left. \frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \right\}. \\
&\exp d_4 \left\{ \frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] - \right. \\
&\quad \left. \frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \right. \\
&\quad \left. \frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \right. \\
&\quad \left. \frac{1}{2} \left(\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \right. \right. \\
&\quad \left. \frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \right) - \left. \right. \\
&\quad \left. \frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \right. \\
&\quad \left. \frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \right\} \\
&= \exp d_4 X. \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \right. \\
&\quad \left. \frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \right\}. \\
&\exp d_4 \left\{ \frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \right. \\
&\quad \frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\
&\quad \frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\
&\quad \frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \left. \right) - \left. \right. \\
&\quad \frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \\
&\quad \frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \left. \right) \\
&\left(\begin{array}{c} \text{By Theorem 26 with} \\ (d_1 + d_2 + d_3) (X + Y) + \\ \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\ d_4 \left\{ Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \right. \\ \left. \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \right. \\ \left. \frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \right\} \\ \left(\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \right) \\ = \frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \\ \frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{array} \right) \left(\begin{array}{c} \end{array} \right)
\end{aligned}$$

We keep on once more.

$$\begin{aligned}
&= \exp d_4 \left\{ -\frac{1}{2} \left(\begin{array}{c} (d_1 + d_2 + d_3) [Y, X] + \frac{1}{2} (d_1 + d_2 + d_3)^2 [[X, Y], X] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], X] \end{array} \right) - \right. \\
&\quad \left. \frac{1}{6} \left(\begin{array}{c} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ \frac{1}{2} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] \end{array} \right) \right\} \\
&\exp d_4 \left\{ X + \frac{1}{2} \left(\begin{array}{c} (d_1 + d_2 + d_3) [Y, X] + \frac{1}{2} (d_1 + d_2 + d_3)^2 [[X, Y], X] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], X] \end{array} \right) + \right. \\
&\quad \left. \frac{1}{6} \left(\begin{array}{c} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ \frac{1}{2} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] \end{array} \right) \right\} \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \right. \\
&\quad \left. \frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \right\} \\
&\exp d_4 \left\{ \begin{array}{c} \frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ \frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ \frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \\ \frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{array} \right\} \\
&= \exp d_4 \left\{ -\frac{1}{2} \left(\begin{array}{c} (d_1 + d_2 + d_3) [Y, X] + \frac{1}{2} (d_1 + d_2 + d_3)^2 [[X, Y], X] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], X] \end{array} \right) - \right. \\
&\quad \left. \frac{1}{6} \left(\begin{array}{c} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ \frac{1}{2} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] \end{array} \right) \right\} \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ X + Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \right. \\
&\quad \left. \frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \right\} \\
&\exp d_4 \left\{ \begin{array}{c} \frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ \frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ \frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \\ \frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{array} \right\} \\
&\left. \begin{array}{c} \text{By Theorem 31 with} \\ (d_1 + d_2 + d_3) (X + Y) + \\ \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\ Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \\ \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ \frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ \frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \end{array} \right) (X) \\
&= X + \frac{1}{2} \left(\begin{array}{c} (d_1 + d_2 + d_3) [Y, X] + \frac{1}{2} (d_1 + d_2 + d_3)^2 [[X, Y], X] + \\ \frac{1}{12} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], X] \end{array} \right) + \\
&\quad \frac{1}{6} \left(\begin{array}{c} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ \frac{1}{2} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] \end{array} \right)
\end{aligned}$$

We keep on once more.

$$\begin{aligned}
&= \exp d_4 \left\{ \begin{aligned} &-\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], X] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &-\frac{1}{2} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] - \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \end{aligned} \right\} + \left\{ \begin{aligned} &-\frac{1}{2} (d_1 + d_2 + d_3) [Y, X] - \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], X] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ &-\frac{1}{2} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] - \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \end{aligned} \right\}. \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ \begin{aligned} &X + Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \end{aligned} \right\}. \\
&\exp d_4 \left\{ \begin{aligned} &\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right\} - \left\{ \begin{aligned} &\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right\}
\end{aligned}$$

We keep on once more.

$$\begin{aligned}
&= \exp d_4 \left\{ \begin{aligned} &-\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], X] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &-\frac{1}{2} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] - \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \end{aligned} \right\} + \left\{ \begin{aligned} &\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\ &d_4 \left\{ \begin{aligned} &X + Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \\ &\frac{1}{2} (d_1 + d_2 + d_3) [Y, X] - \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], X] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] \end{aligned} \right\} \\ &\exp d_4 \left\{ \begin{aligned} &\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right\} \end{aligned} \right\} \\
&\left(\begin{aligned} &\text{By Theorem 31 with} \\ &\delta^{\text{righttt}}(\exp) \left(\begin{aligned} &(d_1 + d_2 + d_3) (X + Y) + \\ &\frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\ &d_4 \left\{ \begin{aligned} &X + Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] \end{aligned} \right\} \end{aligned} \right) \\ &\left(\begin{aligned} &-\frac{1}{2} (d_1 + d_2 + d_3) [Y, X] - \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], X] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] \\ &= -\frac{1}{2} (d_1 + d_2 + d_3) [Y, X] - \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], X] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ &-\frac{1}{2} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] - \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \end{aligned} \right) - \end{aligned} \right)
\end{aligned}$$

We keep on once more.

$$\begin{aligned}
&= \exp d_4 \left\{ \begin{aligned} &-\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], X] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &\frac{1}{2} \left(\begin{aligned} &-\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] - \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \end{aligned} \right) + \end{aligned} \right\} \\
&\exp d_4 \left\{ \begin{aligned} &\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \end{aligned} \right\} \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ \begin{aligned} &X + Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \\ &\frac{1}{2} (d_1 + d_2 + d_3) [Y, X] - \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], X] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] \end{aligned} \right\} \\
&\exp d_4 \left\{ \begin{aligned} &\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ &\frac{1}{2} \left(\begin{aligned} &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right) - \end{aligned} \right\}
\end{aligned}$$

We keep on once more.

$$\begin{aligned}
&= \exp d_4 \left\{ \frac{1}{2} \left(\begin{aligned} &-\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], X] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &-\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], X]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] - \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \end{aligned} \right) + \right\} \\
&\exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ \begin{aligned} &X + Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \\ &\frac{1}{2} (d_1 + d_2 + d_3) [Y, X] - \frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], X] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] \end{aligned} \right\} \\
&\exp d_4 \left\{ \begin{aligned} &\frac{1}{24} (d_1 + d_2 + d_3)^3 [[X - Y, [X, Y]], Y] - \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^3 [X + Y, [[X, Y], Y]] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] + \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [X, Y]]] \end{aligned} \right\} \\
&\left(\delta^{\text{righttt}} (\exp) \left(\begin{aligned} &\text{By Theorem 31 with} \\ &(d_1 + d_2 + d_3) (X + Y) + \\ &\frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ &\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\ &\left\{ \begin{aligned} &X + Y + \frac{1}{2} (d_1 + d_2 + d_3) [X, Y] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], Y] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] + \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [X, Y]] - \\ &\frac{1}{2} (d_1 + d_2 + d_3) [Y, X] - \\ &\frac{1}{4} (d_1 + d_2 + d_3)^2 [[X, Y], X] - \\ &\frac{1}{6} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] \end{aligned} \right) \\ &\left(\frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] \right) \\ &= \frac{1}{4} (d_1 + d_2 + d_3)^2 [X + Y, [Y, X]] + \\ &\frac{1}{8} (d_1 + d_2 + d_3)^3 [X + Y, [X + Y, [Y, X]]] \end{aligned} \right) \right)
\end{aligned}$$

We keep on once more.

$$\begin{aligned}
&= \exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] + \\
&d_4 \left\{ X + Y + (d_1 + d_2 + d_3) [X, Y] + \frac{1}{4} (d_1 + d_2 + d_3)^2 [X - Y, [X, Y]] \right\} . \\
&\exp d_4 (d_1 + d_2 + d_3)^3 \left\{ \begin{array}{l} \frac{1}{24} [X, [X - Y, [X, Y]] + \\ \left(\frac{1}{12} - \frac{1}{8} \right) [X + Y, [X, [X, Y]] + \\ \left(\frac{1}{12} + \frac{1}{12} - \frac{1}{8} \right) [X + Y, [X + Y, [Y, X]] - \\ \frac{1}{24} [Y, [X - Y, [X, Y]] + \\ \left(\frac{1}{12} - \frac{1}{8} \right) [X + Y, [Y, [X, Y]] - \\ \left(\frac{1}{12} + \frac{1}{12} - \frac{1}{8} \right) [X + Y, [X + Y, [X, Y]] \end{array} \right\} \\
&= \exp (d_1 + d_2 + d_3 + d_4) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [X - Y, [X, Y]] . \\
&\exp d_4 (d_1 + d_2 + d_3)^3 \left\{ \begin{array}{l} -\frac{1}{24} ([X, [Y, [X, Y]] + [Y, [X, [X, Y]]]) + \\ \frac{1}{24} [X + Y, [X + Y, [Y, X]] - \\ -\frac{1}{24} ([X, [Y, [X, Y]] + [Y, [X, [X, Y]]]) - \\ \frac{1}{24} [X + Y, [X + Y, [X, Y]] \end{array} \right\} \\
&= \exp (d_1 + d_2 + d_3 + d_4) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [X - Y, [X, Y]] . \\
&\exp -\frac{1}{12} d_4 (d_1 + d_2 + d_3)^3 \left(\begin{array}{c} [X, [Y, [X, Y]] + [Y, [X, [X, Y]] + \\ [X + Y, [X + Y, [X, Y]] \end{array} \right) \\
&= \exp (d_1 + d_2 + d_3 + d_4) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] + \\
&\frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [X - Y, [X, Y]] - \\
&\frac{1}{48} (d_1 + d_2 + d_3 + d_4)^4 \left(\begin{array}{c} [X, [Y, [X, Y]] + [Y, [X, [X, Y]] + \\ [X + Y, [X + Y, [X, Y]] \end{array} \right)
\end{aligned}$$

■

References

- [1] Baker, H. F.: Alternants and continuous groups, Proceedings of the London Mathematical Society, Second Series, **3** (1905), 24-47.
- [2] Campbell, J. E.: On a law of combination of operators bearing on the theory of continuous transformation groups, Proceedings of the London Mathematical Society, **28** (1897), 381-390.

- [3] Campbell, J. E.: On a law of combination of operators (second paper), Proceedings of the London Mathematical Society, **29** (1898), 14-32.
- [4] Fer, F.: Résolution de l'équation matricielle $dU/dt = pU$ par produit infini d'exponentielles matricielles, Académie Royale de Belgique de la Classe des Sciences, 5e Série, **44** (1958), 818-829.
- [5] Hall, Brian C.: Lie Groups, Lie Algebras, and Representations; an Elementary Introduction, Springer-Verlag, New York, 2003.
- [6] Hausdorff, F.: Die symbolische Exponential Formel in der Gruppentheorie, Berichte über die Verhandlungen Sächsischen Akademie der Wissenschaften zu Leipzig, **58** (1906), 19-48.
- [7] Kock, Anders: Taylor series calculus for ring objects of line type, Journal of Pure and Applied Algebra, **12** (1978), 271-293.
- [8] Kriegl, Andreas and Michor, Peter W.: The Convenient Setting of Global Analysis, American Mathematical Society, Rhode Island, 1997.
- [9] Lavendhomme, René: Basic Concepts of Synthetic Differential Geometry, Kluwer Academic Publishers, Dordrecht, 1996.
- [10] Magnus, Wilhelm: On the exponential solution of differential equations for a linear operator, Communications on Pure and Applied Mathematics, **7** (1954), 649-673.
- [11] Magnus, Wilhelm, Karrass, Abraham and Solitar, Donald: Combinatorial Group Theory, Presentations of Groups in Terms of Generators and Relations, Dover, New York, 1976.
- [12] Michor, Peter W.: Topics in Differential Geometry, American Mathematical Society, Rhode Island, 2008.
- [13] Omori, H., Maeda, Y. and Yoshioka, A.: On regular Fréchet Lie groups I. Some differential geometric expressions of Fourier integral operators on a Riemannian manifolds, Tokyo Journal of Mathematics, **3** (1980), 353-390.
- [14] Omori, H., Maeda, Y. and Yoshioka, A.: On regular Fréchet Lie groups II. Composition rules of Fourier integral operators on a Riemannian manifold, Tokyo Journal of Mathematics, **4** (1981), 221-253.
- [15] Omori, H., Maeda, Y. and Yoshioka, A.: On regular Fréchet Lie groups III, Tokyo Journal of Mathematics, **4** (1981), 255-277.
- [16] Omori, H., Maeda, Y. and Yoshioka, A.: On regular Fréchet Lie groups IV. Definitions and fundamental theorems, Tokyo Journal of Mathematics, **5** (1982), 365-398.
- [17] Omori, H., Maeda, Y. and Yoshioka, A.: On regular Fréchet Lie groups V. Several basic properties, Tokyo Journal of Mathematics, **6** (1983), 39-64.

- [18] Omori, H., Maeda, Y., Yoshioka, A. and Kobayashi, O.: On regular Fréchet Lie groups VI. Infinite dimensional Lie groups which appear in general relativity, *Tokyo Journal of Mathematics*, **6** (1983), 217-246.
- [19] Varadarajan, V.S.: *Lie Groups, Lie Algebras, and their Representations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1974.
- [20] Zassenhaus, Hans: Über Lieschen Ringe mit Primzahlcharakteristik, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, **13** (1939), 1-100.